Quiz !

1. Favorite Food = ______.
2. Finish the definition:
   A limit point x of a set $A \subseteq \mathbb{R}$ is ...
3. State the Bolzano-Weierstrass Thm.
4. Give the sequential definition of "compact set".

Connected Sets & Disconnected Sets.

Given a set $A \subseteq \mathbb{R}$, we say that $\{A_1, A_2\}$ is a separation of A if $A = A_1 \cup A_2$, $A_1 \cap A_2 = \emptyset$, $A_1 \neq \emptyset$, $A_2 \neq \emptyset$, (i.e. A_1 & A_2 are disjoint), and

$\exists\, U_1, U_2$ open $U_1 \cap U_2 = \emptyset$ s.t.

$$A_1 = A \cap U_1$$
$$A_2 = A \cap U_2.$$

picture

We say A is <u>disconnected</u> if $\exists$ a separation $\{A_1, A_2\}$ of A.

A set is <u>connected</u> if it is not disconnected, i.e. there does not exist a separation of A.

<u>Lemma</u>: Intervals in $\mathbb{R}$ are connected.

e.g. $(1,\infty)$, $[2,7)$, $(-\infty,+3]$, $(1,2)$.

<u>Thm</u>. The only connected sets in $\mathbb{R}$ are intervals.

New Topic: Functional limits & Continuity.

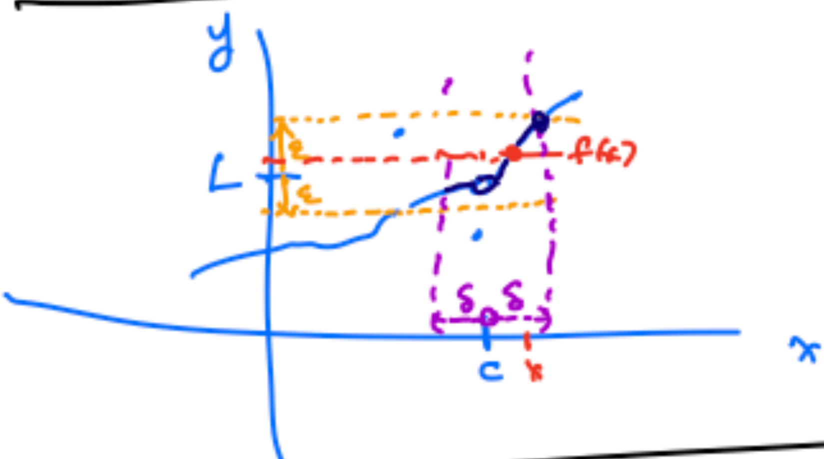
Definition Let $f : A \to \mathbb{R}$ be a function, where $A \subseteq \mathbb{R}$. Let c be a limit point of A.

We say $\lim_{x \to c} f(x) = L \iff$

$\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$\forall x \in A$ s.t. $0 < |x - c| < \delta$, so that $x \neq c$

then $|f(x) - L| < \varepsilon$.



Exercise 4.2.5. Use Definition 4.2.1 to supply a proper proof for the following limit statements.

(a) $\lim_{x\to 2}(3x+4) = 10$.

(b) $\lim_{x\to 0} x^3 = 0$.

(c) $\lim_{x\to 2}(x^2+x-1) = 5$.

(d) $\lim_{x\to 3} 1/x = 1/3$.

ⓐ $\lim\limits_{x\to 2}(3x+4) = 10$

Scratch work: Domain of $f(x) = 3x+4$ is $\mathbb{R}$.

Goal: $|f(x) - L| < \varepsilon$

$|3x+4-10| < \varepsilon$

Need $|x-2| < \underset{\delta}{\bigcirc}$

$\Leftrightarrow |3x-6| < \varepsilon$

$|3(x-2)| < \varepsilon$

$|3||x-2| < \varepsilon$

$3|x-2| < \varepsilon$

$|x-2| < \frac{\varepsilon}{3}$. $\leftarrow \delta$

Pf. $\forall \varepsilon > 0$, let $\delta = \frac{\varepsilon}{3} > 0$. Then,

if $0 < |x-2| < \delta = \frac{\varepsilon}{3}$, then $3|x-2| < \varepsilon$

$$\Rightarrow |3x-6| < \varepsilon \Rightarrow |3x+4-10| < \varepsilon.$$

Thus $\lim_{x\to 2} 3x+4 = 10.$ ∎

(b) $\lim_{x\to 0} x^3 = 0$

Scratch: Domain of x^3 is $\mathbb{R}$.

Want $|x^3 - 0| < \varepsilon \Leftarrow$ Need $|x-0| < \delta$ ok

$\Leftrightarrow |x^3| < \varepsilon$

$\Downarrow$

$|x|^3 < \varepsilon$

$\sqrt[3]{x}$ increasing $\Rightarrow |x| < \varepsilon^{1/3}$ δ.

Pf. $\forall \varepsilon > 0$, let $\delta = \varepsilon^{1/3} > 0$.

Then, if $0 < |x-0| < \delta = \varepsilon^{1/3}$,

$$|x|^3 < \delta^3 = \varepsilon \quad \left(\text{since } x^3 \text{ is increasing}\right)$$

$$\Rightarrow |x^3 - 0| < \varepsilon.$$

$\therefore \lim_{x\to 0} x^3 = 0.$ ∎

ⓒ $\lim_{x \to 2} (x^2 + x - 1) = 5$

Scratch work:

Want: $|x^2+x-1-5| < \varepsilon$

Need $|x-2| < \delta$ $\Longrightarrow$

$\Updownarrow$

$|x^2+x-6| < \varepsilon$

$|(x+3)(x-2)| < \varepsilon$

$|x+3|\,|x-2| < \varepsilon$

$|x+3|\,|x-2| < \underline{6}\,\underline{|x-2| < \varepsilon}$ $\Leftarrow$ Need $\delta \le 1$

If $|x-2|<1$

$1<x<3$

$4<x+3<6$

$4<|x+3|<6$ ✓

$|x-2| < \frac{\varepsilon}{6}$ δ

Let $\delta = \min\{1, \frac{\varepsilon}{6}\}$.

Pf. $\forall \varepsilon > 0$, let $\delta = \min\{1, \frac{\varepsilon}{6}\} > 0$.

Then, if $0 < |x-2| < \delta = \min\{1, \frac{\varepsilon}{6}\}$,

then $|x-2| < \frac{\varepsilon}{6}$

$$\Rightarrow \quad 6|x-2| < \varepsilon$$

Since $|x-2|<1$, $2-1<x<2+1 \Rightarrow 1<x<3$

$$\Rightarrow 4<x+3<6.$$

$$\therefore \quad \underset{*_0}{|x+3|\,|x-2|} < 6\underset{*_0}{|x-2|} < \varepsilon$$

$$\Rightarrow \quad |(x+3)(x-2)| < \varepsilon$$

$$\Rightarrow |x^2+x-6| < \varepsilon$$

$$\Rightarrow |x^2+x-1-5| < \varepsilon.$$

Thus, $\lim_{x\to 2} x^2+x-1 = 5$. ▨

(d) Prove $\lim_{x\to 3} \frac{1}{x} = \frac{1}{3}$.

Scratch: Domain of $\frac{1}{x}$: $\mathbb{R}\setminus\{0\}$.

$0<|x-3|<\delta \Rightarrow$ Need $\left|\frac{1}{x}-\frac{1}{3}\right|<\varepsilon$

$$\left|\frac{3-x}{3x}\right| < \varepsilon$$

$$\Leftrightarrow \frac{|3-x|}{3|x|} < \varepsilon$$

$$\frac{1}{3|x|}|x-3| = \frac{|x-3|}{3|x|} \leq \frac{1}{6}|x-3| < \varepsilon$$

set $\delta \leq 1$

$0 < |x-3| < 1$

$|x-3| < 1$

$3-1 < x < 3+1$

$2 < x < 4$

$6 < 3x < 12$

$\frac{1}{6} > \frac{1}{3x} > \frac{1}{12}$

$\frac{1}{3|x|}$

$|x-3| < 6\varepsilon$

We will need

$\delta = \min\{1, 6\varepsilon\}$.

Pf: $\forall \varepsilon > 0$, let $\delta = \min\{1, 6\varepsilon\}$.

Then, if $0 < |x-3| < \delta = \min\{1, 6\varepsilon\}$,

then $|x-3| < 6\varepsilon$

$\Rightarrow \frac{1}{6}|x-3| < \varepsilon$

Since $|x-3| < 1$,

$3-1 < x < 3+1$

$2 < x < 4$

$6 < 3x < 12$

$\frac{1}{6} > \frac{1}{3x} = \frac{1}{3|x|} > \frac{1}{12}$.

Then

$\frac{1}{3|x|}|x-3| < \varepsilon$

$$\Rightarrow \left|\frac{x-3}{3x}\right| < \varepsilon$$

$$\Rightarrow \left|\frac{3-x}{3x}\right| < \varepsilon$$

$$\Rightarrow \left|\frac{1}{x} - \frac{1}{3}\right| < \varepsilon.$$

Thus, $\lim_{x\to 3} \frac{1}{x} = \frac{1}{3}$. $\blacksquare$

Equivalent versions of the defn of functional limit:

<u>Thm</u> The following are equivalent:
Let $f: A \to \mathbb{R}$, c be a limit point of A.

① $\lim_{x\to c} f(x) = L$

② (Topological version) For every $\varepsilon > 0$, $\exists\, \delta > 0$ s.t. $f\left(\left(V_\delta(c) \cap A\right) \setminus \{c\}\right) \subseteq V_\varepsilon(L)$.

$f(\overset{\text{set}}{S}) = \{f(x) : x \in S\}$.

③ (Sequential Criterion)

For every sequence (x_n) s.t.

$x_n \to c$, $x_n \in A \setminus \{c\}$ $\forall n \in \mathbb{N}$,

we have $f(x_n) \to L$.

Algebraic Limit Theorem for Functional Limits. (ALTFL)

Suppose $\lim_{x \to c} f(x) = L$ and

$\lim_{x \to c} g(x) = M$.

Then ⓐ $\lim_{x \to c} \left(f(x) \pm g(x) \right) = L \pm M$.

ⓑ $\lim_{x \to c} \left(f(x) \cdot g(x) \right) = L \cdot M$.

ⓒ $\lim_{x \to c} \left(\frac{f(x)}{g(x)} \right) = \frac{L}{M}$,

if $g(x) \neq 0$ near $x = c$ and $M \neq 0$.

ⓓ $\lim_{x \to c} \sqrt{f(x)} = \sqrt{L}$

if $f(x) \geq 0$ near $x = c$, $L \geq 0$.

OLTFL: If $f(x) \leq g(x)$ $\forall x \in A$

~~and~~ c is a limit pt of A,

and if $\lim_{x\to c} f(x) = L$

$\lim_{x\to c} g(x) = M$.

Then $L \leq M$.

Special Limits: $\cancel{K} \in \mathbb{R}$

- $\lim_{x\to c} K = K$. (any δ works)
- $\lim_{x\to c} x = c$ (let $\delta = \varepsilon$)

"Corollary" - limits of rational functions & fcns involving square roots of rational fcns are what you think.
